

A note on density of M -sets in geometric progression *

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Abstract

Let M be a set of positive integers, and let $\mu(M)$ denote the maximal density among sets of nonnegative integers in which no two elements have difference belonging to M . Motzkin presented the problem of determining $\mu(M)$. This problem is completely settled when $|M| \leq 2$ and some partial results are known for several families of M when $|M| \geq 3$. Recently, Pandey and Tripathi investigated this quantity when M is related to arithmetic progressions. In this note, as a corollary, we determine $\mu(M)$ when M is a geometric progression.

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1 Introduction

Let \mathbb{N} be the set of all nonnegative integers. For a positive real number x and $S \subseteq \mathbb{N}$, we denote by $S(x)$ the number of elements $n \in S$ such that $n \leq x$.

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The upper and lower densities of S , denoted by $\overline{\delta}(S)$ and $\underline{\delta}(S)$ respectively, are given by

$$\overline{\delta}(S) := \limsup_{x \rightarrow \infty} \frac{S(x)}{x}, \quad \underline{\delta}(S) := \liminf_{x \rightarrow \infty} \frac{S(x)}{x}.$$

If $\overline{\delta}(S) = \underline{\delta}(S)$, we denote the common value by $\delta(S)$, and say that S has density $\delta(S)$.

Given a set of positive integers M , we call a set $S \subseteq \mathbb{N}$ is an M -set if $a \in S, b \in S$ implies $a - b \notin S$. In an unpublished problem collection, T. S. Motzkin [13] posed the problem of determining the quantity

$$\mu(M) := \sup_S \overline{\delta}(S),$$

where the supremum is taken over all M -sets S . In [3], Cantor and Gordon proved that if $|M| = 1$, $\mu(M) = 1/2$ and that if $M = \{m_1, m_2\}$, then $\mu(M) = [(m_1 + m_2)/2]/(m_1 + m_2)$. Some partial results for the general case are also presented. Later, Haralambis [9] determined $\mu(M)$ for most members of the families $\{1, j, k\}$ and $\{1, 2, j, k\}$. In 1999, Gupta and Tripathi [7] completely determined $\mu(M)$ when M is a finite arithmetic progression. In 2011, Pandey and Tripathi [16] investigated this quantity when M is related to arithmetic progressions. For related results, one may refer to [8], [14] and [15].

Recently, Chen and Yang [6], Khovanova and Konyagin [10] studied the maximal density among sets of nonnegative integers in which no two elements have quotient belonging to M .

Motzkin's problem also has connections with some other problems, such as the T -colouring problem, problems related to the fractional chromatic number of distance graphs and the Lonely Runner Conjecture. One can refer to [1], [2], [4], [5], [11], [12] and [17].

In this note, we determine $\mu(M)$ when M is a geometric progression. In fact, we prove the following more general result.

Theorem 1. *Let $M = \{m_1, m_2, \dots\}$, where $m_1 < m_2 < \dots$ and $m_1 | m_i$ for all integers $i \geq 1$. Let $m_2/m_1 = q$. If $m_i/m_1 \equiv \pm 1 \pmod{q+1}$ for all*

integers $i \geq 1$, then we have

$$\mu(M) = \begin{cases} 1/2 & \text{if } q \text{ is odd;} \\ \frac{q}{2(q+1)} & \text{if } q \text{ is even.} \end{cases}$$

From Theorem 1 we obtain the following corollaries immediately.

Corollary 1. *For any subset C of $\{q^i(q+2)^j : i \geq 0, j \geq 0\}$ and $M = C \cup \{1, q\}$, where q is an integer with $q \geq 2$, we have $\mu(M) = 1/2$ if q is odd, and $\mu(M) = \frac{q}{2(q+1)}$ if q is even.*

Corollary 2. *If $M = \{a, aq, \dots, aq^n\}$, where a, q, n are positive integers with $q \geq 2$, then $\mu(M) = 1/2$ if q is odd, and $\mu(M) = \frac{q}{2(q+1)}$ if q is even.*

2 Proofs

Lemma 1. (See [3, Theorem 1].) *Let $M = \{m_1, m_2, m_3, \dots\}$, and let c and m be positive integers such that $(c, m) = 1$. Put*

$$d = \min_k |cm_k|_m,$$

where $|x|_m$ denotes the absolute value of the absolutely least remainder of x (mod m). Then $\mu(M) \geq d/m$.

Lemma 2. (See [3, Theorem 2].) *Let $M_1 = \{m_1, m_2, \dots\}$ and $M_2 = \{dm_1, dm_2, \dots\}$, where d is a positive integer. Then $\mu(M_1) = \mu(M_2)$.*

Proof of Theorem 1. By $m_2/m_1 = q$, $m_1|m_i$ ($i \geq 1$) and Lemma 2, we may assume that $m_1 = 1$ and $m_2 = q$.

Suppose that q is odd. For any M -set S , by $m_1 = 1$, it contains no two consecutive integers. Thus, $\mu(M) \leq 1/2$. On the other hand, by $m_i \equiv \pm 1 \pmod{q+1}$ and $2 \mid q+1$, we have M consists of only odd numbers. Hence, the example $S = \{0, 2, 4, \dots\}$ shows that equality can hold, and so $\mu(M) = 1/2$.

Now we consider the case in which q is even. If $\mu(M) > \frac{q}{2(q+1)}$, then there exists an M -set S and an interval $[c, c+q]$ such that $|S \cap [c, c+q]| > q/2$.

That is, $|S \cap [c, c + q]| \geq q/2 + 1$. Noting that $1 \in M$, we have S does not contain consecutive integers, and so $S = [c, c + 2, \dots, c + q]$. It follows that $q \in S - S$, a contradiction. Hence, $\mu(M) \leq \frac{q}{2(q+1)}$.

Next we shall prove that $\mu(M) \geq \frac{q}{2(q+1)}$. Since

$$\frac{q}{2} \times m_i \equiv \pm \frac{q}{2} \pmod{q+1}$$

for all positive integers i , by Lemma 1, taking $c = q/2$ and $m = q + 1$, we have $(c, m) = 1$ and $d = q/2$. Thus, $\mu(M) \geq \frac{q}{2(q+1)}$.

Therefore, $\mu(M) = 1/2$ if q is odd and $\mu(M) = \frac{q}{2(q+1)}$ if q is even. \square

Remark 1. *If q is even and large enough, then the difference of S missing M will degenerate into missing 1. This is consistent with $\lim_{q \rightarrow \infty} \frac{q}{2(q+1)} = 1/2$.*

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